Centralizers and Normalizers (D+F 2.2)

Let G be a group and $A \subseteq G$ a subset (not necessarily a subgroup.

Def: The <u>centralizer</u> of A, denoted $C_G(A)$, is the set of elements of G that commute w/ all elements of A.

i.e. $$C_G(A) := \{g \in G \mid gag^{-1} = a \ \forall \ a \in A\}.$$

(Note that $gag^{-1} = a \iff ga = ag$.)

Claim: $C_G(A) \leq G$.

Pf: $1 \in C_G(A)$, so it's nonempty.

If $x, y \in C_G(A)$, we want to show $xy^{-1} \in C_G(A)$ as well:

Let $a \in A$. Then $ya = ay \Rightarrow a = y^{-1}ay$

$$\Rightarrow xax^{-1} = x(y^{-1}ay)x^{-1}$$
$$\Rightarrow a = (xy^{-1})a(yx^{-1}) = (xy^{-1})a(xy^{-1})^{-1}$$

Thus, $xy^{-1} \in C_G(A)$, so it's a subgroup. $\square$

Ex: 1.) $C_G(1) = G$, since everything commutes w/ the identity.

2.) From the homework, if $n = 2k$,

then $C_{D_{2n}}(r^k) = D_{2n}$ and $C_{D_{2n}}(D_{2n}) = \{1, r^k\}$.

The subgroup $C_G(G)$ is the set of elements that commute with every element of G, and is denoted $Z(G)$. It is called the <u>center</u> of G.

Note that $Z(G) = G \iff G$ is abelian.

<u>Ex</u>: From the homework, if $n \geq 3$ then

- If $n = 2k$, then $Z(D_{2n}) = \{1, r^k\}$
- If n is odd, $Z(D_{2n}) = 1$.

<u>Def</u>: Define $gAg^{-1} = \{gag^{-1} \mid a \in A\}$. The <u>normalizer</u> of A in G is the set $N_G(A) = \{g \in G \mid gAg^{-1} = A\}$

(It will be clear soon why it's called the normalizer and why it's useful.)

The normalizer is also a subgroup of G (proof is nearly identical to the one for the centralizer).

Notice: Being in the normalizer of a set is weaker than being in its centralizer. That is,

If $g \in C_G(A)$ then $gag^{-1} = a \quad \forall \ a \in A$ so

$gAg^{-1} = A$. Thus $g \in N_G(A) \implies C_G(A) \leq N_G(A)$.

Ex: Consider $S_3 = \{1, (12), (13), (23), (1\,2\,3), (1\,3\,2)\}$

Let $A = \{1, (12)\}$.

What is $C_{S_3}(A)$? Well, by Lagrange's Theorem (see HW), $|C_{S_3}(A)| \mid 6$. But also $2 \mid |C_{S_3}(A)|$ since $A \leq C_{S_3}(A)$.

So $|C_{S_3}(A)| = 2$ or 6, but $(1\,2)(1\,3) \neq (1\,3)(1\,2)$ (They send 1 different places), so $C_{S_3}(A) = A$.

For the normalizer of A, $\sigma \in N_{S_3}(A)$ iff

$$\{\sigma(1\,2)\sigma^{-1}, \underbrace{\sigma 1 \sigma^{-1}}_{= 1}\} = \{(1\,2), 1\}.$$

$\iff \sigma(1\,2)\sigma^{-1} = (1\,2)$. But this would imply $\sigma \in C_{S_3}(A)$,

so $N_{S_3}(A) = C_{S_3}(A)$.

Ex: Consider D_8 again. Let $A = \{1, r, r^2, r^3\}$.

By Lagrange's Theorem, since $sr \neq rs$, $|C_{D_8}(A)| = 4$, so $C_{D_8}(A) = A$.

However, this means $N_{D_8}(A) = C_{D_8}(A)$ or D_8.

$sr^i s = s^2 r^{-i} = r^{-i} \in A$, so $s \in N_{D_8}(A)$, so $N_{D_8}(A) = D_8$.